Discrete kinetic theory and hyperbolic balance laws (Workshop on the Boltzmann Equation, Microlocal Analysis and Related Topics)

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Dedicated to Professor Yoshinori Morimoto

By

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Abstract

The discrete Boltzmann equation is an important example of hyperbolic balance laws. We first review the basic concepts in discrete kinetic theory. Then, as a generalization of the H-function, we introduce the notion of mathematical entropy for hyperbolic balance laws and discuss the symmetrization of the system. Also we introduce Craftsmanship Condition and under this condition we discuss the dissipative structure of the system. Finally, we review the results on the global existence and asymptotic decay of solutions.

§ 1. Introduction

The discrete Boltzmann equation is a model system of the Boltzmann equation with a set of discrete velocites, and is given in the form:

$$F_{i,t} + v_i \cdot \nabla_x F_i = Q_i(F, F), \qquad i = 1, \cdots, m, \tag{1.1}$$

where $F_i = F_i(t, x)$ denotes the mass density of gas particles with the velocity $v_i = (v_{i1}, \cdots, v_{in}) \in \mathbb{R}^n$ (constant vector) at time t and position $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$; $Q_i(F, F)$ is the quadratic term related to binary collisions, whose explicit form is given in the next section.

On the other hand, hyperbolic balance laws are systems of partial differential equations of the form

$$w_t + \sum_{j=1}^{n} f^j(w)_{x_j} = g(w). \tag{1.2}$$

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Here w is the unknown m-vector valued function of time t and space variable $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, f^j $(j = 1, \cdots, n)$ and g are given m-vector valued smooth functions of $w \in \mathcal{O}_w$, and $\mathcal{O}_w$ is a convex open set in $\mathbb{R}^m$. Notice that the discrete Boltzmann equation (1.1) is a typical example of hyperbolic balance laws.

In this paper we review the basic concepts in discrete kinetic theory for (1.1). They are collision invariant, Maxwellian, the Boltzmann H-function and the corresponding H-theorem. Then, as a generalization of the H-function, we introduce the notion of mathematical entropy for hyperbolic balance laws (1.2). We observe that the symmetrization of the system (1.2) is equivalent to the existence of a mathematical entropy. Also we introduce Craftsmanship Condition and under this condition we discuss the dissipative structure of the system (1.2). Finally, we show the global existence and asymptotic decay of solutions.

§ 2. Discrete kinetic theory

The collision term $Q_i(F, F)$ in (1.1) is given explicitly by

$$Q_i(F,G) = \frac{1}{2} \sum_{j,k,\ell=1}^{m} \big\{ A_{k\ell}^{ij}(F_k G_\ell + F_\ell G_k) - A_{ij}^{k\ell}(F_i G_j + F_j G_i) \big\}, \tag{2.1}$$

where $A_{k\ell}^{ij}$ are nonnegative constants satisfying $A_{k\ell}^{ij} \neq 0$ for some i, j, k, ℓ, and

$$A_{\ell k}^{ij} = A_{k\ell}^{ij} = A_{k\ell}^{ji}, \qquad A_{k\ell}^{ij} = A_{ij}^{k\ell} \qquad \text{for all} \quad i, j, k, \ell = 1, \cdots, m. \tag{2.2}$$

We write $F = (F_i)$, $Q(F,G) = (Q_i(F,G))$ and $V^j = \mathrm{diag}(v_{ij})$; F and $Q(F,G)$ are m-vectors such that $Q(F,G)$ is bi-linear with respect to F and G, and V^j are $m \times m$ diagonal matrices. Then the discrete Boltzmann equation (1.1) is written in the vector form as

$$F_t + \sum_{j=1}^{n} V^j F_{x_j} = Q(F,F). \tag{2.3}$$

Following [4, 7], we introduce the basic concepts in discrete kinetic theory.

Definition 2.1 (Collision invariant [4, 7]). A vector $\phi = (\phi_i) \in \mathbb{R}^m$ is called a collision invariant if $A_{k\ell}^{ij}(\phi_i + \phi_j - \phi_k - \phi_\ell) = 0$ for all $i, j, k, \ell = 1, \cdots, m$.

We denote by $\mathcal{M}$ the set of collision invariants. Then $\mathcal{M}$ is a subspace of $\mathbb{R}^m$ such that $0 < \dim \mathcal{M} < m$; notice that $(1, \cdots, 1)^T \in \mathcal{M}$, where the superscript T denotes the transpose.

Proposition 2.2 (Characterization of collision invariant [4, 7]). *Let $\phi \in \mathbb{R}^m$. Then the following three conditions are equivalent to each other.*

(a) *ϕ is a collision invariant, i.e., $\phi \in \mathcal{M}$.*

(b) *$\langle \phi, \, Q(F,G) \rangle = 0$ for any $F, G \in \mathbb{R}^m$.*

(c) *$\langle \phi, \, Q(F,F) \rangle = 0$ for any $F \in \mathbb{R}^m$.*

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^m$

By a technical computation, using (2.1) and (2.2), we find that

$$\begin{aligned}\langle \phi, \, Q(F,G) \rangle &= \sum_i \phi_i Q_i(F,G) \\ &= \frac{1}{8} \sum_{i,j,k,\ell} A^{ij}_{k\ell} (\phi_i + \phi_j - \phi_k - \phi_\ell)(F_k G_\ell + F_\ell G_k - F_i G_j - F_j G_i)\end{aligned} \tag{2.4}$$

for any $\phi, F, G \in \mathbb{R}^m$. The above proposition is an easy consequence of this identity.

We denote by $\mathbb{R}^m_+$ the convex open set of vectors $F = (F_i) \in \mathbb{R}^m$ satisfying $F_i > 0$ for all $i = 1, \cdots, m$.

Definition 2.3 (Maxwellian [4, 7]). A vector $F = (F_i) \in \mathbb{R}^m_+$ is called a Maxwellian if $A^{ij}_{k\ell}(F_i F_j - F_k F_\ell) = 0$ for all $i, j, k, \ell = 1, \cdots, m$.

Proposition 2.4 (Characterization of Maxwellian [4, 7]). *Let $F = (F_i) \in \mathbb{R}^m_+$. Then the following four conditions are equivalent to each other.*

(a) *F is a Maxwellian.*

(b) *$\log F = (\log F_i) \in \mathbb{R}^m$ is a collision invariant, i.e., $\log F \in \mathcal{M}$.*

(c) *$Q(F,F) = 0$.*

(d) *$\langle \log F, \, Q(F,F) \rangle = 0$.*

The identity (2.4) plays a crucial role also in the proof of this proposition.

Definition 2.5 (Boltzmann H-function [4, 7]). For a vector $F = (F_i) \in \mathbb{R}^m_+$, the following function is called the Boltzmann H-function.

$$H(F) := \langle \log F, \, F \rangle = \sum_{i=1}^m F_i \log F_i. \tag{2.5}$$

If $F = (F_i) \in \mathbb{R}^m_+$ is a solution to the discrete Boltzmann equation (1.1), then the H-function satisfies

$$\begin{aligned}&\Big(\sum_i F_i \log F_i\Big)_t + \sum_i v_i \cdot \nabla_x \big(F_i \log F_i\big) \\ &= -\frac{1}{4} \sum_{i,j,k,\ell} A^{ij}_{k\ell} (F_i F_j - F_k F_\ell) \log \big(F_i F_j / F_k F_\ell\big).\end{aligned} \tag{2.6}$$

To see this equality, we multiply (1.1) by $1 + \log F_i$ and add for $i = 1, \cdots, m$. Then, using (2.4) with $\phi_i = 1 + \log F_i$ and $G = F$, we obtain (2.6). In vector notation, (2.6) is written as

$$H(F)_t + \sum_{j=1}^{n} J^j(F)_{x_j} = \langle \log F,\, Q(F,F)\rangle, \tag{2.7}$$

where $J^j(F) := \langle \log F,\, V^j F\rangle$ is the flux corresponding to the H-function $H(F)$. We observe that the right hand side of (2.6) is non-positive for any $F \in \mathbb{R}^m_+$. Namely, $\langle \log F,\, Q(F,F)\rangle \le 0$ for any $F \in \mathbb{R}^m_+$. Moreover, the equality $\langle \log F,\, Q(F,F)\rangle = 0$ holds if and only if F is a Maxwellian by Proposition 2.4. Thus we may say that the H-function $H(F)$ is decreasing in time t and F converges to a Mawellian as $t \to \infty$. This is called the *H-theorem* in discrete kinetic theory.

Finally in this section, we linearize the discrete Boltzmann equation (2.3) around a constant Maxwellian $M = (M_i) \in \mathbb{R}^m_+$. Letting $\Lambda_M = \mathrm{diag}\,(M_i)$, we put

$$F = M + \Lambda_M f. \tag{2.8}$$

Then the equation (2.3) is transformed into

$$\Lambda_M f_t + \sum_{j=1}^{n} V^j_M f_{x_j} + L_M f = \Gamma_M(f,f), \tag{2.9}$$

where $V^j_M = V^j \Lambda_M = \mathrm{diag}\,(v_{ij} M_i)$ and

$$L_M f = -2Q(M, \Lambda_M f), \qquad \Gamma_M(f,g) = Q(\Lambda_M f, \Lambda_M g). \tag{2.10}$$

Proposition 2.6 (Properties of L_M and Γ_M [4, 7])**.** *Let $M = (M_i) \in \mathbb{R}^m_+$ be a Maxwellian. Then we have:* (i) *L_M is real symmetric and nonnegative definite such that its kernel coincides with $\mathcal{M}$.* (ii) *$\Gamma_M(f,g)$ is bi-linear with respect to $f, g \in \mathbb{R}^m$ and satisfies $\Gamma_M(f,g) \in \mathcal{M}^\perp$ for any $f, g \in \mathbb{R}^m$.*

By virtue of this proposition we see that the transformed equation (2.9) is a symmetric hyperbolic system with symmetric relaxation. To show Proposition 2.6, we compute

$$\begin{aligned}\langle f,\, L_M g\rangle &= -2\langle f,\, Q(M, \Lambda_M g)\rangle \\ &= \frac{1}{8} \sum_{i,j,k,\ell} A^{ij}_{k\ell}(M_i M_j + M_k M_\ell)(f_i + f_j - f_k - f_\ell)(g_i + g_j - g_k - g_\ell),\end{aligned}$$

where we used (2.4) with $\phi = f$, $F = M$ and $G = \Lambda_M g$ and the fact that $A^{ij}_{k\ell}(M_i M_j - M_k M_\ell) = 0$ for all $i, j, k, \ell = 1, \cdots, m$, i.e., M is a Maxwellian. This identity proves (i) of Proposition 2.6, while (ii) of Proposition 2.6 follows from Proposition 2.2.

§ 3. Hyperbolic balance laws

As a generalization of the Boltzmann H-function in discrete kinetic theory, we introduce the notion of a mathematical entropy for hyperbolic balance laws (1.2). To this end, we set

$$\begin{aligned} &\mathcal{M} := \{\phi \in \mathbb{R}^m;\ \langle \phi,\, g(w)\rangle = 0 \ \text{ for any } w \in \mathcal{O}_w\}, \\ &\mathcal{E} := \{w \in \mathcal{O}_w;\ g(w) = 0\}, \end{aligned} \tag{3.1}$$

where $\mathcal{O}_w$ is a convex open set in $\mathbb{R}^m$. Then $\mathcal{M}$ is a subspace of $\mathbb{R}^m$ and we see that $g(w) \in \mathcal{M}^\perp$ for any $w \in \mathcal{O}_w$. Also $\mathcal{E}$ is the set of equilibrium states for (1.2). In discrete kinetic theory, $\mathcal{M}$ is called the space of collision invariants, and $\mathcal{E}$ denotes the set of Mawellians.

Definition 3.1 (Mathematical entropy [11]). Let $\eta = \eta(w)$ be a smooth function defined in a convex open set $\mathcal{O}_w$. Then $\eta(w)$ is called a mathematical entropy for hyperbolic balance laws (1.2) if the following four statements hold:
(a) $\eta(w)$ is strictly convex in $\mathcal{O}_w$ in the sense that the Hessian $D_w^2\eta(w)$ is positive definite for $w \in \mathcal{O}_w$.
(b) $D_w f^j(w)(D_w^2\eta(w))^{-1}$ is symmetric for $w \in \mathcal{O}_w$ and $j = 1, \cdots, n$.
(c) Let $w \in \mathcal{O}_w$. Then $w \in \mathcal{E}$ holds if and only if $(D_w\eta(w))^T \in \mathcal{M}$.
(d) For $w \in \mathcal{E}$, the matrix $-D_w g(w)(D_w^2\eta(w))^{-1}$ is symmetric and nonnegative definite such that its null space coincides with $\mathcal{M}$.

The notion of mathematical entropy was first introduced by Godunov [5] and by Friedrichs and Lax [3] for hyperbolic systems of conservation laws (including the compressible Euler equation as an example). Then in 1988, this notion was extended by Kawashima and Shizuta [10] (cf. [8, 9]) to hyperbolic-parabolic systems of conservation laws (including the compressible Navier-Stokes equation as an example). The above definition of mathematical entropy for hyperbolic balance laws was first formulated by Kawashima and Yong [11] in 2004 (cf. [2]). You can check that the Boltzmann H-function for the discrete Boltzmann equation satisfies all the statements of the above definition.

Next we introduce the notion of symmetrization. Let $w = w(u)$ be a diffeomorphism from an open set $\mathcal{O}_u$ onto $\mathcal{O}_w$. Then (1.2) can be transformed into

$$A^0(u)u_t + \sum_{j=1}^n A^j(u)u_{x_j} = h(u), \tag{3.2}$$

where

$$
\begin{aligned}
&A^0(u) = D_u w(u), \\
(3.3) \qquad &A^j(u) = D_u f^j(w(u)) = D_w f^j(w(u)) D_u w(u), \\
&h(u) = g(w(u)), \qquad L(u) = -D_u g(w(u)) = -D_w g(w(u)) D_u w(u).
\end{aligned}
$$

Definition 3.2 (Symmetric dissipative system [11]). The system (3.2) is called symmetric dissipative if the following four statements hold:
(a) $A^0(u)$ is symmetric and positive definite for $u \in \mathcal{O}_u$.
(b) $A^j(u)$ is symmetric for $u \in \mathcal{O}_u$ and $j = 1, \cdots, n$.
(c) Let $u \in \mathcal{O}_u$. Then $h(u) = 0$ holds if and only if $u \in \mathcal{M}$.
(d) For $u \in \mathcal{M}(\cap \mathcal{O}_u)$, the matrix $L(u)$ is symmetric and nonnegative definite such that its null space coincides with $\mathcal{M}$.

As in [5, 3, 10] for hyperbolic (hyperbolic-parabolic) systems of conservation laws, we can show that the symmetrization of hyperbolic balance laws is characterized by the existence of a mathematical entropy.

Theorem 3.3 (Mathematical entropy and symmetrization [11]). *The following two statements are equivalent.*
(i) *Hyperbolic balance laws* (1.2) *has a mathematical entropy* $\eta = \eta(w)$.
(ii) *There is a diffeomorphism* $w = w(u)$ *by which* (1.2) *is transformed to a symmetric dissipative system* (3.2).

Proof. We give a short outline of the proof. Suppose that the system (1.2) has a mathematical entropy $\eta = \eta(w)$. We define the mapping $u = u(w)$ by

$$(3.4) \qquad u = u(w) := (D_w \eta(w))^T.$$

This mapping $u = u(w)$ is a diffeomorphism from the convex open set $\mathcal{O}_w$ onto an open set $\mathcal{O}_u$ and satisfies $D_w u(w) = D_w^2 \eta(w)$. Let $w = w(u)$ be the corresponding inverse mapping. Then this $w = w(u)$ is a diffeomorphism from $\mathcal{O}_u$ onto $\mathcal{O}_w$ satisfying $D_u w(u) = (D_w u(w))^{-1} = (D_w^2 \eta(w))^{-1}$. By this diffeomorphism $w = w(u)$, the system (1.2) can be transformed into a symmetric dissipative system (3.2), where

$$
\begin{aligned}
&A^0(u) = D_u w(u) = (D_w^2 \eta(w))^{-1}, \\
(3.5) \qquad &A^j(u) = D_w f^j(w(u)) D_u w(u) = D_w f^j(w) (D_w^2 \eta(w))^{-1}, \\
&h(u) = g(w(u)), \qquad L(u) = -D_w g(w(u)) D_u w(u) = -D_w g(w) (D_w^2 \eta(w))^{-1}.
\end{aligned}
$$

This shows that (i) implies (ii).

Conversely, we suppose that there is a diffeomorphism $w = w(u)$, from an open set $\mathcal{O}_u$ onto the convex open set $\mathcal{O}_w$, by which the system (1.2) is transformed to a symmetric dissipative system (3.2). We see that $\mathcal{O}_u$ is simply connected as $\mathcal{O}_w$ is convex. Also, $D_u w(u) = A^0(u)$ is symmetric. Therefore the Poincaré lemma ensures the existence of a smooth function $\tilde{\eta} = \tilde{\eta}(u)$ such that $D_u \tilde{\eta}(u) = w(u)^T$. By using this $\tilde{\eta}(u)$, we put

$$\eta(w) = \langle u(w),\, w\rangle - \tilde{\eta}(u(w)), \tag{3.6}$$

where $u = u(w)$ is the inverse mapping of $w = w(u)$. Then this $\eta = \eta(w)$ becomes a mathematical entropy for (1.2), which satisfies $D_w \eta(w) = u(w)^T$ and $D_w^2 \eta(w) = (D_u w(u))^{-1} = A^0(u)^{-1}$. Thus we have verified that (ii) implies (i). For the detailed proof we refer the reader to [11]. □

Finally in this section, we derive the equation satisfied by the mathematical entropy for hyperbolic balance laws (1.2). Let $\eta = \eta(w)$ be a mathematical entropy for (1.2) and define $u = u(w)$ by (3.4). Then we observed that (1.2) is transformed to a symmetric dissipative system (3.2) by the diffeomorphism $w = w(u)$ which is the inverse mapping of $u = u(w)$. Since each $D_u f^j(w(u)) = A^j(u)$ is symmetric, the Poincaré lemma ensures the existence of a smooth function $\tilde{q}^j = \tilde{q}^j(u)$ satisfying $D_u \tilde{q}^j(u) = f^j(w(u))^T$, where $j = 1, \cdots, n$. We put

$$q^j(w) = \langle u(w),\, f^j(w)\rangle - \tilde{q}^j(u(w)), \qquad j = 1, \cdots, n. \tag{3.7}$$

Then this $q^j(w)$ turns out to be the corresponding entropy flux. In fact, a simple computation using (3.4) shows that

$$D_w q^j(w) = D_w \eta(w) D_w f^j(w), \qquad j = 1, \cdots, n. \tag{3.8}$$

Consequently, taking the inner product of (1.2) with $u(w) = (D_w \eta(w))^T$ and using (3.8), we arrive at

$$\eta(w)_t + \sum_{j=1}^{n} q^j(w)_{x_j} = \langle (D_w \eta(w))^T,\, g(w)\rangle. \tag{3.9}$$

This is the equation satisfied by the mathematical entropy $\eta = \eta(w)$ and is regarded as a generalization of (2.7) for the Boltzmann H-function in discrete kinetic theory.

Concerning the right hand side of (3.9), we have the following result.

Proposition 3.4 ([11]). *Let $\bar{u} \in \mathcal{M}(\cap\, \mathcal{O}_u)$ be a state which is corresponding to a given $\bar{w} \in \mathcal{E}$. Then there are positive constants δ and c such that*

$$\langle (D_w \eta(w))^T,\, g(w)\rangle = \langle u,\, h(u)\rangle \le -c|(I-P)u|^2$$

for any $u \in \mathcal{O}_u$ with $|u - \bar{u}| \le \delta$, where P denotes the orthogonal projection onto $\mathcal{M}$.

This proposition implies that the right hand side of (3.9) is non-positive at least in a neighborhood of $\bar{u} \in \mathcal{M}$. Namely, we have $\langle (D_w \eta(w))^T, g(w) \rangle \leq 0$ in a neighborhood of $\bar{u} \in \mathcal{M}$. Moreover, the equality $\langle (D_w \eta(w))^T, g(w) \rangle = 0$ holds if and only if $Pu = u$, i.e., $u \in \mathcal{M}$, which means $w \in \mathcal{E}$. Thus, similarly to the H-theorem in discrete kinetic theory, we may say that the mathematical entropy $\eta(w)$ is decreasing in time t and w converges to an equilibrium state as $t \to \infty$. This is corresponding to the second law in thermodynamics if the physical entropy S is defined by $S = -\eta(w)$.

§ 4. Dissipative structure

Let $\bar{u} \in \mathcal{M}(\cap \mathcal{O}_u)$ be a constant state and consider the linearized system of (3.2) at $u = \bar{u}$:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{4.1}$$

where $A^0 = A^0(\bar{u})$, $A^j = A^j(\bar{u})$ and $L = L(\bar{u})$. Notice that A^0 is real symmetric and positive definite, A^j are real symmetric, and L is real symmetric and nonnegative definite such that its null space coincides with $\mathcal{M}$. We take the Fourier transform of (4.1) with respect to $x \in \mathbb{R}^n$. This gives

$$A^0 \hat{u}_t + i|\xi| A(\omega) \hat{u} + L\hat{u} = 0, \tag{4.2}$$

where $A(\omega) := \sum_{j=1}^{n} A^j \omega_j$ with $\omega = \xi/|\xi| \in S^{n-1}$ (the unit sphere in $\mathbb{R}^n$).

Following [14, 13], we review the general theory on the dissipative structure of the system (4.1). First, we formulate a structural condition which yields the standard decay estimate of solutions to the system (4.1).

Craftsmanship Condition ([14])**:** There is a $m \times m$ matrix $K(\omega)$ depending smoothly on $\omega \in S^{n-1}$ with the following properties: $K(-\omega) = -K(\omega)$ for $\omega \in S^{n-1}$,
(i) $K(\omega)A^0$ is skew-symmetric for $\omega \in S^{n-1}$,
(ii) $(K(\omega)A(\omega))^{sy} + L$ is positive definite for $\omega \in S^{n-1}$, where X^{sy} denotes the symmetric part of the matrix X.

Theorem 4.1 (Linear decay [14])**.** *Suppose that the system* (4.1) *satisfies Craftsmanship Condition. Then the solution u of* (4.1) *with the initial data u_0 satisfies the following pointwise estimate in the Fourier space:*

$$|\hat{u}(t,\xi)| \leq Ce^{-c\rho(\xi)t} |\hat{u}_0(\xi)|, \tag{4.3}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. Moreover the solution u satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2} \|u_0\|_{L^1} + Ce^{-ct} \|\partial_x^k u_0\|_{L^2} \tag{4.4}$$

for $k \geq 0$. Here C and c are positive constants.

The pointwise estimate (4.3) can be proved by the energy method for the system (4.2), which makes use of the matrix $K(\omega)$ in Craftsmanship Condition. We construct a Lyapunov function $E(\hat{u})$ by

$$E(\hat{u}) = \langle A^0\hat{u},\, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle, \tag{4.5}$$

where α is a small positive constant, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^m$. Then we can derive the energy inequality

$$E(\hat{u})_t + c\rho(\xi)|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0, \tag{4.6}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$, and c is a positive constant. Since $E(\hat{u})$ is equivalent to $|\hat{u}|^2$, the energy inequality (4.6) yields the desired estimate (4.3). The decay estimate (4.4) is an easy consequence of (4.3).

Next we give the characterization of the dissipative structure for the system (4.1). To this end, following [13], we formulate the following structural condition.

Stability Condition ([13])**:** Let $\varphi \in \mathbb{R}^m$ satisfy $L\varphi = 0$ (i.e., $\varphi \in \mathcal{M}$) and $\mu A^0\varphi + A(\omega)\varphi = 0$ for some $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$. Then $\varphi = 0$.

Also we consider the eigenvalue problem for (4.2). Let $\lambda = \lambda(i\xi)$ be the eigenvalues of (4.2):

$$\lambda A^0\varphi + i|\xi|A(\omega)\varphi + L\varphi = 0, \tag{4.7}$$

where $\varphi \in \mathbb{C}^m$ with $\varphi \neq 0$. Note that the eigenvalues λ solve the characteristic equation $\det(\lambda A^0 + i|\xi|A(\omega) + L) = 0$. The dissipativity of the system (4.1) is completely characterized as follows.

Theorem 4.2 (Characterization of dissipative structure [13])**.** *The following four statements for* (4.1) *are equivalent.*
(a) *Stability Condition.*
(b) *Craftsmanship Condition.*
(c) *Uniform dissipativity, i.e.,* $\mathrm{Re}\,\lambda(i\xi) \le -c\rho(\xi)$ *for* $\xi \in \mathbb{R}^n$*, where* $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$*, and c is a positive constant.*
(d) *Strict dissipativity, i.e.,* $\mathrm{Re}\,\lambda(i\xi) < 0$ *for* $\xi \in \mathbb{R}^n$ *with* $\xi \neq 0$.

We observed that the energy method for (4.2) in the Fourier space, which makes use of Craftsmanship Condition, yields the energy inequality (4.6). If we apply the same computation to (4.7), then we obtain

$$\mathrm{Re}\,\lambda\, E(\varphi) + c\rho(\xi)|\varphi|^2 + c|(I-P)\varphi|^2 \le 0,$$

where $E(\varphi)$ is defined in (4.5). Since $E(\varphi)$ is equivalent to $|\varphi|^2$, the above inequality gives the estimate $\mathrm{Re}\,\lambda \leq -c\rho(\xi)$. This shows that (b) implies (c) in Theorem 4.2. Also, that (c) implies (d) is trivial. On the other hand, to show that (d) $\Rightarrow$ (a), we suppose that there exists a $\varphi \in \mathbb{R}^m$ with $\varphi \neq 0$ such that $L\varphi = 0$ and $\mu A^0\varphi + A(\omega)\varphi = 0$ hold for some $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$. Then we see that

$$i|\xi|\mu A^0\varphi + i|\xi|A(\omega)\varphi + L\varphi = 0.$$

This shows that $\lambda := i|\xi|\mu$ is an eigenvalue of (4.2) satisfying $\mathrm{Re}\,\lambda = 0$, which is a contradiction to (d). Thus we see that (d) implies (a). On the other hand, the proof of (a) $\Rightarrow$ (b) is not so easy and we refer the reader to [13] for the details.

§ 5. Global existence and decay

The global existence and decay of solutions to the initial value problem for hyperbolic balance laws (1.2) was proved by many authors by assuming the existence of "suitable entropy function" and Craftsmanship Condition. We refer the reader to [1, 6, 12, 18]. Here, following [12], we review the results on the global existence and asymptotic decay of solutions to (1.2).

Theorem 5.1 (Global existence [12]). *Suppose that* (1.2) *possesses a mathematical entropy and the corresponding symmetric dissipative system* (3.2) *satisfies Stability Condition at $\bar{u} \in \mathcal{M}$, where $\bar{u}$ is the constant state corresponding to a given constant state $\bar{w} \in \mathcal{E}$. Let $n \geq 1$ and suppose that the initial data w_0 satisfy $w_0 - \bar{w} \in H^s$ with $s > n/2+1$. Then if $E_0 = \|w_0 - \bar{w}\|_{H^s}$ is small, the initial value problem for* (1.2) *has a unique global solution w with $w - \bar{w} \in C([0,\infty); H^s)$. The solution satisfies the uniform estimate*

$$\|w(t) - \bar{w}\|_{H^s}^2 + \int_0^t \|(I-P)u(\tau)\|_{H^s}^2 + \|\partial_x w(\tau)\|_{H^{s-1}}^2 \, d\tau \leq CE_0^2$$

for $t \geq 0$, where C is a positive constant. Here u is defined in (3.4) *and P is the orthogonal projection onto $\mathcal{M}$. Moreover, the solution w converges to the constant state $\bar{w}$ for $t \to \infty$, namely,*

$$\|\partial_x^l (w(t) - \bar{w})\|_{L^\infty} \longrightarrow 0$$

as $t \to \infty$, where $0 \leq l \leq s - s_0$ with $s_0 = [n/2]+1$.

This theorem can be proved by the energy method which makes use of the strict convexity of the mathematical entropy and the matrix $K(\omega)$ in Craftsmanship Condition. In particular, we use the following energy form to derive the L^2 energy estimate for $w - \bar{w}$:

$$\mathcal{H}(w) = \eta(w) - \eta(\bar{w}) - \langle \bar{u},\, w - \bar{w} \rangle, \tag{5.1}$$

where $\eta(w)$ is the mathematical entropy. Since $\eta(w)$ is strictly convex in w and $u = (D_w\eta(w))^T$, $\mathcal{H}(w)$ is equivalent to the quadratic function $|w-\bar{w}|^2$ and hence to $|u-\bar{u}|^2$. By using (3.9) and (1.2), we know that the energy form $\mathcal{H}(w)$ satisfies

$$\mathcal{H}(w)_t + \sum_{j=1}^{n} \mathcal{Q}^j(w)_{x_j} = \langle u,\, g(w)\rangle, \tag{5.2}$$

where $u = (D_w\eta(w))^T$ and $\mathcal{Q}^j(w) = q^j(w) - q^j(\bar{w}) - \langle \bar{u},\, f^j(w) - f^j(\bar{w})\rangle$. By virtue of Proposition 3.4, you can expect that the integration of (5.2) with respect to $x \in \mathbb{R}^n$ and t gives the desired L^2 estimate for $w - \bar{w}$.

Next we remark the following decay estimate that can be obtained by applying the time-weighted energy method.

Theorem 5.2 (Decay: case of L^2 data [12])**.** *Let $n \geq 2$ and $s > n/2+1$. Under the same conditions of Theorem 5.1 we have the following decay estimates:*

$$\|\partial_x^k(w(t) - \bar{w})\|_{L^2} \leq CE_0(1+t)^{-k/2}$$

for $0 \leq k \leq s$, and

$$\|(I-P)\partial_x^k u(t)\|_{L^2} \leq CE_0(1+t)^{-(k+1)/2}$$

for $0 \leq k \leq s-1$, where C is a positive constant.

When the initial data are also in L^1, we can apply the semigroup approach based on the linear decay result in Theorem 4.1 together with the above time-weighted energy method and obtain the following sharp decay estimate.

Theorem 5.3 (Decay: case of L^1 data [12])**.** *Let $n \geq 1$, and $s > n/2+1$ for $n \geq 2$ and $s \geq 3$ for $n = 1$. Under the same conditions of Theorem 5.1, we suppose that $w_0 - \bar{w} \in H^s \cap L^1$ and $E_1 = \|w_0 - \bar{w}\|_{H^s\cap L^1}$ is small. Then the global solution w to* (1.2) *satisfies the following decay estimates:*

$$\|\partial_x^k(w(t) - \bar{w})\|_{L^2} \leq CE_1(1+t)^{-n/4-k/2}$$

for $0 \leq k \leq s-1$, and

$$\|(I-P)\partial_x^k u(t)\|_{L^2} \leq CE_1(1+t)^{-n/4-(k+1)/2}$$

for $0 \leq k \leq s-2$, where C is a positive constant.

All these previous results assumed that the initial data are in the Sobolev space H^s with $s > s_c := n/2+1$. This regularity assumption seems to be essential even for the local existence result because our system (1.2) or (3.2) is a quasi-linear hyperbolic system and any additional regularity can not be expected. In the analysis we need the property $\partial_x u \in L^\infty$ for solutions $u \in H^s$, and this requires the regularity $s > s_c := n/2 + 1$. Recently, this regularity assumption was weakened by using the Besov space. Namely, we have a similar global existence result for initial data in the Besov space $B^{s_c}_{2,1}$ with the critical regularity $s_c := n/2 + 1$, and this generalization is based on the fact that $B^{n/2}_{2,1} \subset L^\infty$. Also, the L^1 assumption on the initial data for sharp decay estimate in Theorem 5.3 was weakened by using another Besov space $\dot{B}^{-n/2}_{2,\infty}$, where we know that $L^1 \subset \dot{B}^{-n/2}_{2,\infty}$. For the details of these generalizations, we refer the reader to [15, 16, 17].

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